Verification of Chase-Lev work-stealing deque (work in progress)

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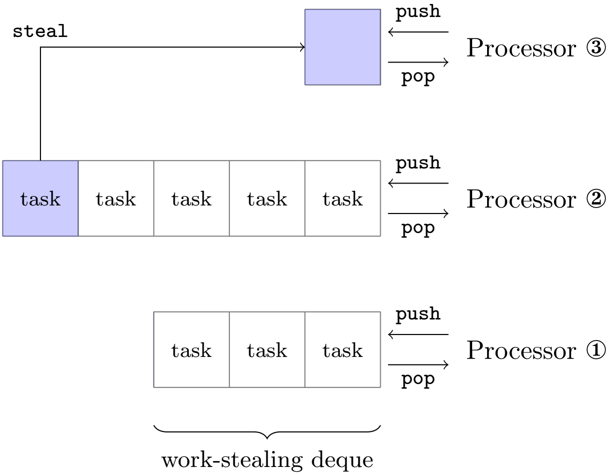
Context: scheduler for task-based parallelism

- Cilk (C, C++)
- Threading Building Blocks (C++)
- Taskflow (C++)
- Tokio (Rust)
- Goroutines (Go)
- Domainslib (OCaml 5)

Work-stealing



Chase-Lev work-stealing deque

1. *The Implementation of the Cilk-5 Multithreaded Language.* Frigo, Leiserson & Randall (1998).
   - lock
2. *Thread Scheduling for Multiprogrammed Multiprocessors.* Arora, Blumofe & Plaxton (1998).
   - non-blocking
   - one fixed size array, potential overflow
3. *A dynamic-sized nonblocking work stealing deque.* Hendler, Lev, Moir & Shavit (2004).
   - non-blocking
   - list of small arrays, no overflow
4. *Dynamic circular work-stealing deque.* Chase & Lev (2005).
   - non-blocking
   - circular arrays, no overflow

The rest of this talk

- Specification in Iris (logically atomic triples).
- Sketch of the proof for a *simplified* version with one infinite array (as opposed to multiple circular arrays).
  - physical state
  - logical state
  - external future-dependent linearization point
- Why we need prophecy variables.

Specification

Physical state

Logical state

Prophecy variables

Specification — `chaselev_make`

$$\frac{\{\,\text{True}\,\}}{\dfrac{\texttt{chaselev\_make ()}}{\left\{\lambda\, t.\ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-model}\ t\ []\ \ast\ \text{chaselev-owner}\ t\right\}}}$$

Specification — `chaselev_make`

$$\frac{\{\,\text{True}\,\}}{\frac{\texttt{chaselev\_make ()}}{\left\{\lambda\, t.\ \text{chaselev-inv}\ t\ \iota\ *\ \text{chaselev-model}\ t\ []\ *\ \text{chaselev-owner}\ t\right\}}}$$

t is an instance of Chase-Lev deque.
Enforces a protocol (using an Iris invariant).

Specification — `chaselev_make`

$$\frac{\{\,\text{True}\,\}\quad \texttt{chaselev\_make ()}}{\left\{\,\lambda\, t.\ \text{chaselev-inv}\ t\ \iota\ *\ \text{chaselev-model}\ t\ []\ *\ \text{chaselev-owner}\ t\,\right\}}$$

Asserts the list of values that t logically contains.

Specification — `chaselev_make`

$$\frac{\{\,\text{True}\,\}}{\frac{\texttt{chaselev\_make ()}}{\left\{\,\lambda\, t.\ \text{chaselev-inv}\ t\ \iota\ *\ \text{chaselev-model}\ t\ []\ *\ \text{chaselev-owner}\ t\,\right\}}}$$

Gives the owner exclusive access to his end of t.

Specification — `chaselev_push`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_push}\ t\ v,\ \uparrow\iota$$

$$\left\langle\ \exists.\ \text{chaselev-model}\ t\ (vs +\!\!+ [v])\ \right\rangle$$

$$\left\{\ ().\ \text{chaselev-owner}\ t\ \right\}$$

Specification — `chaselev_push`

Specification of a concurrent operation ($\simeq$ transaction):
standard triple + logically atomic triple

$$\frac{\{\, P \,\}}{\frac{\langle\, \forall\, \overline{x}.\, P_{\text{lin}} \,\rangle}{\frac{e,\ \mathcal{E}}{\frac{\langle\, \exists\, \overline{y}.\, Q_{\text{lin}} \,\rangle}{\{\, res.\, Q \,\}}}}}$$

P : private precondition

Q : private postcondition

P_{lin} : public precondition

Q_{lin} : public postcondition

Specification — `chaselev_push`

For a concurrent data structure:

$$\frac{\frac{\frac{\frac{\{\, \text{???-inv} \cdots * P \,\}}{\langle \forall\, \overline{x}.\ \text{???-model} \cdots \rangle}}{e,\ \mathcal{E}}}{\langle \exists\, \overline{y}.\ \text{???-model} \cdots \rangle}}{\{\, res.\, Q \,\}}$$

Specification — `chaselev_push`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_push}\ t\ v,\ \uparrow\iota$$

$$\left\langle\ \exists.\ \text{chaselev-model}\ t\ (vs +\!\!+ [v])\ \right\rangle$$

$$\left\{\ \texttt{()}.\ \text{chaselev-owner}\ t\ \right\}$$

t is an instance of Chase-Lev deque.

Specification — `chaselev_push`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_push}\ t\ v,\ \uparrow\iota$$

$$\left\langle\ \exists.\ \text{chaselev-model}\ t\ (vs +\!\!+ [v])\ \right\rangle$$

$$\left\{\ \texttt{()}.\ \text{chaselev-owner}\ t\ \right\}$$

This operation is reserved to the owner of t.

Specification — `chaselev_push`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ *\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_push}\ t\ v,\ \uparrow\iota$$

$$\left\langle\ \exists.\ \text{chaselev-model}\ t\ (vs +\!\!+ [v])\ \right\rangle$$

$$\left\{\ \texttt{()}.\ \text{chaselev-owner}\ t\ \right\}$$

v is atomically pushed at the owner's end of t.

Specification — `chaselev_pop`

$$\dfrac{\dfrac{\dfrac{\left\{\ \text{chaselev-inv}\ t\ \iota\ *\ \text{chaselev-owner}\ t\ \right\}}{\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle}}{\texttt{chaselev\_pop}\ t,\ \uparrow\iota}}{\dfrac{\left\langle\ \exists\, o.\ \bigvee\begin{bmatrix} vs = [\,] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = vs' +\!\!+ [v] * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ vs' \end{bmatrix}\ \right\rangle}{\left\{\ o.\ \ \text{chaselev-owner}\ t\ \right\}}}$$

Specification — `chaselev_pop`

$$\dfrac{\left\{\ \text{chaselev-inv}\ t\ \iota\ \ *\ \ \text{chaselev-owner}\ t\ \right\}}{\dfrac{\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ \ vs\ \right\rangle}{\dfrac{\texttt{chaselev\_pop}\ t,\ \uparrow \iota}{\dfrac{\left\langle\ \exists\, o.\ \bigvee \begin{bmatrix} vs = [\,] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = vs' +\!\!+ [v] * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ vs' \end{bmatrix}\ \right\rangle}{\left\{\ o.\ \ \text{chaselev-owner}\ t\ \right\}}}}}$$

t is an instance of Chase-Lev deque.

Specification — `chaselev_pop`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ *\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_pop}\ t,\ \uparrow \iota$$

$$\left\langle\ \exists\, o.\ \bigvee \left[\begin{array}{l} vs = [\,] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = vs' +\!\!+ [v] * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ vs' \end{array}\right]\ \right\rangle$$

$$\left\{\ o.\ \ \text{chaselev-owner}\ t\ \right\}$$

This operation is reserved to the owner of t.

Specification — `chaselev_pop`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ *\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_pop}\ t,\ \uparrow\iota$$

$$\left\langle\ \exists\, o.\ \bigvee\left[\begin{array}{l} vs = [] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [] \\ \exists\, v, vs'.\, vs = vs' +\!\!+ [v] * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ vs' \end{array}\right]\ \right\rangle$$

$$\left\{\ o.\ \ \text{chaselev-owner}\ t\ \right\}$$

Either 1) t is seen empty
or 2) some value v is atomically popped at the owner's end of t.

Specification — `chaselev_steal`

$$\frac{\left\{\ \text{chaselev-inv}\ t\ \iota\ \right\}}{\dfrac{\left\langle \forall\, vs.\ \ \text{chaselev-model}\ t\ vs \right\rangle}{\dfrac{\texttt{chaselev\_steal}\ t,\ \uparrow\iota}{\dfrac{\left\langle \exists\, o.\ \bigvee \begin{bmatrix} vs = [] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [] \\ \exists\, v, vs'.\ vs = v :: vs' * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ vs' \end{bmatrix} \right\rangle}{\{\, o.\ \text{True}\,\}}}}}$$

Specification — `chaselev_steal`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \right\}$$

$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ \ vs\ \right\rangle$$

$$\texttt{chaselev\_steal}\ t,\ \uparrow\iota$$

$$\left\langle\ \exists\, o.\ \ \bigvee\left[\begin{array}{l} vs = [\,] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = v :: vs' * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ \ vs' \end{array}\right]\ \right\rangle$$

$$\{\, o.\ \text{True}\,\}$$

t is an instance of Chase-Lev deque.

Specification — `chaselev_steal`

$$
\begin{array}{c}
\left\{\ \text{chaselev-inv}\ t\ \iota\ \right\} \\
\hline
\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle \\
\hline
\texttt{chaselev\_steal}\ t,\ \uparrow\iota \\
\hline
\left\langle\ \exists\, o.\ \bigvee \left[\begin{array}{l} vs = [\,] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = v :: vs' * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ vs' \end{array} \right]\ \right\rangle \\
\hline
\{\, o.\ \text{True}\,\}
\end{array}
$$

Either 1) t is seen empty
or 2) some value v is atomically popped at the thieves' end of t.

Specification

Physical state

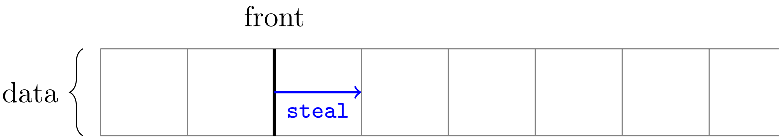
Logical state

Prophecy variables

Physical state

data: infinite array storing all values

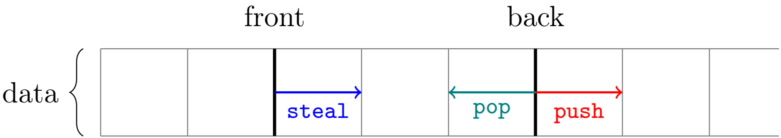
Physical state



data: infinite array storing all values

front: *monotone* index for thieves' end

Physical state

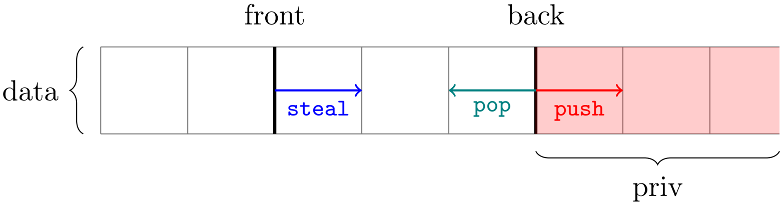


data: infinite array storing all values

front: *monotone* index for thieves' end

back: index for owner's end

Physical state



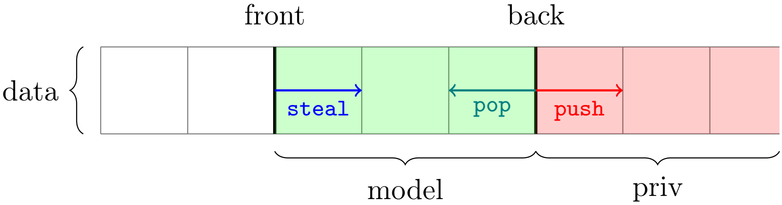
data: infinite array storing all values

front: *monotone* index for thieves' end

back: index for owner's end

priv: list of private values (controlled by owner)

Physical state



data: infinite array storing all values

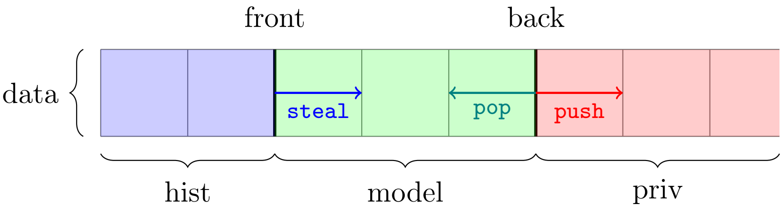
front: *monotone* index for thieves' end

back: index for owner's end

priv: list of private values (controlled by owner)

model: list of contained values

Physical state



data: infinite array storing all values

front: *monotone* index for thieves' end

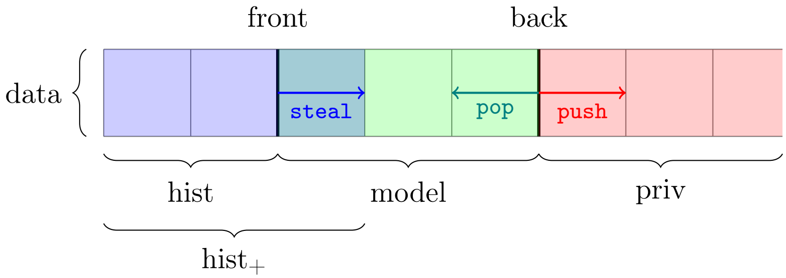
back: index for owner's end

priv: list of private values (controlled by owner)

model: list of contained values

hist: *monotone* list of history values

Physical state



data: infinite array storing all values

front: *monotone* index for thieves' end

back: index for owner's end

priv: list of private values (controlled by owner)

model: list of contained values

hist: *monotone* list of history values

hist_+: *monotone* list of extended history values

Specification

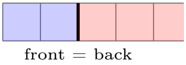
Physical state

Logical state

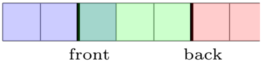
Prophecy variables

Logical state

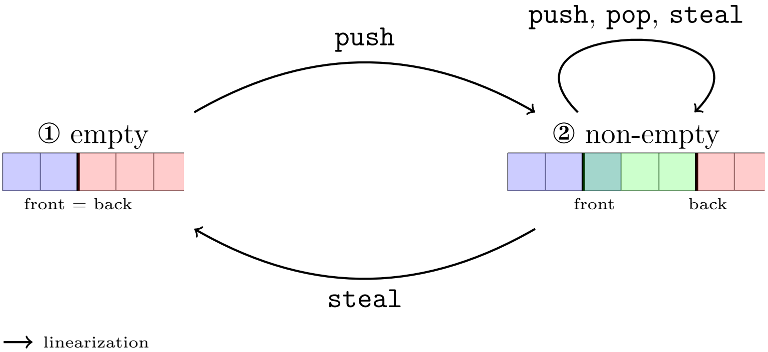
① empty



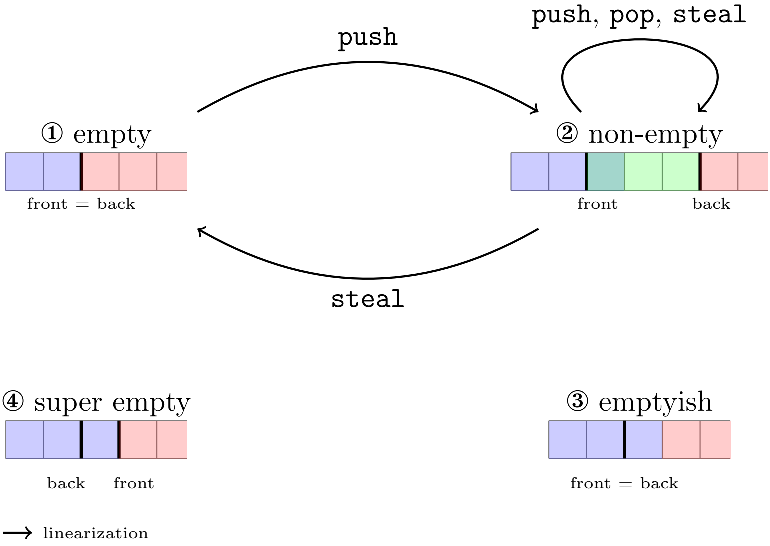
② non-empty



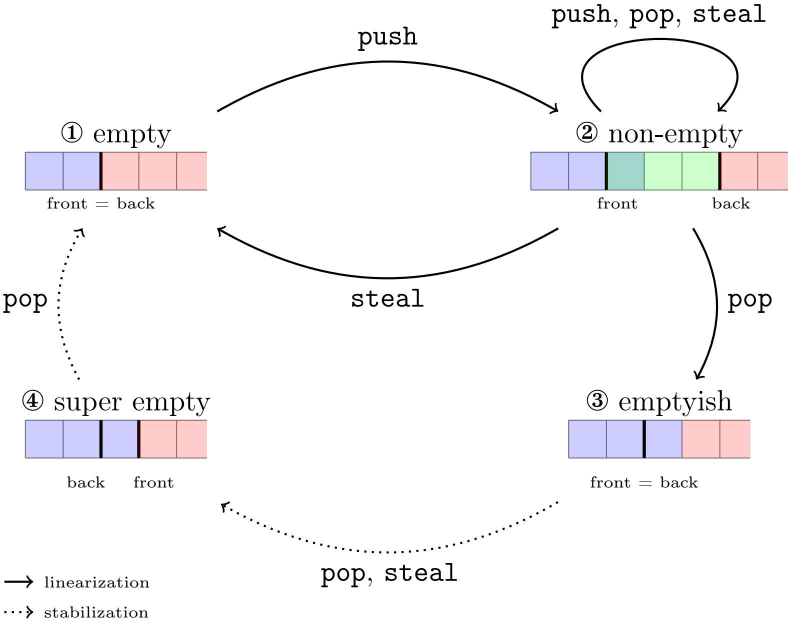
Logical state



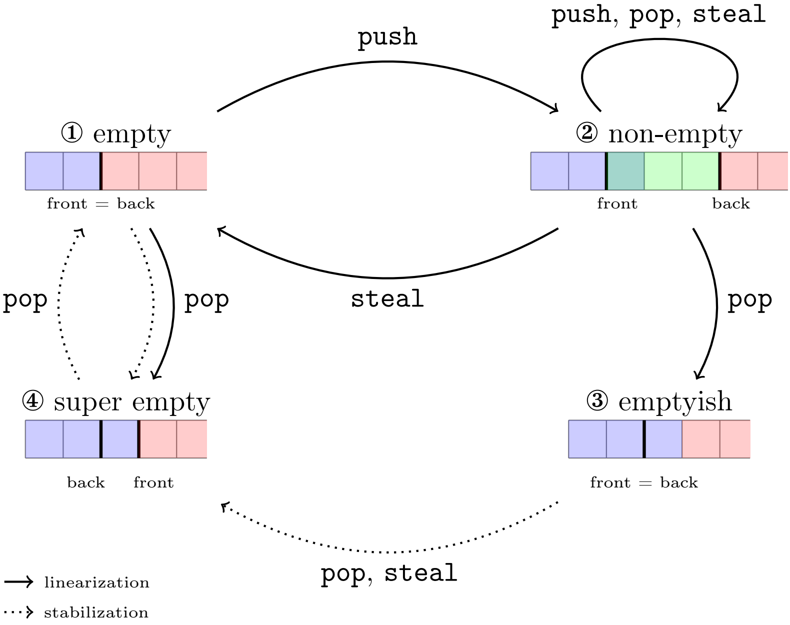
Logical state



Logical state



Logical state



Specification

Physical state

Logical state

Prophecy variables

Prophecy variables

The future is ours: prophecy variables in separation logic.
Jung, Lepigre, Parthasarathy, Rapoport, Timany, Dreyer & Jacobs (2020).

$$\{\,\text{True}\,\}\ \texttt{NewProph}\ \{\,\lambda\, p.\, \exists\, \mathit{prophs}.\, \text{proph}\ p\ \mathit{prophs}\,\}$$

$$\frac{\text{atomic } e \qquad \text{proph } p\ \mathit{prophs} \qquad \text{WP } e \left\{ \begin{array}{l} \lambda w.\, \forall\, \mathit{prophs}'. \\ \mathit{prophs} = (w, v) :: \mathit{prophs}' \mathrel{-\!\!*} \\ \text{proph } p\ \mathit{prophs}' \mathrel{-\!\!*} \\ \Phi\ w \end{array} \right\}}{\text{WP } \texttt{Resolve}\ e\ p\ v\ \{\,\Phi\,\}}$$

Prophecy variables with memory

$$\{\,\text{True}\,\}\ \texttt{NewProph}\ \{\,\lambda\, p.\, \exists\, \gamma.\, \mathit{prophs}.\ \text{proph}\ p\ \gamma\ []\ \mathit{prophs}\,\}$$

$$\dfrac{\begin{array}{c}\text{atomic } e \\ \text{proph}\ p\ \gamma\ \mathit{past}\ \ \mathit{prophs} \\ \text{WP}\ e\ \left\{\begin{array}{l}\lambda w.\, \forall\, \mathit{prophs}'. \\ \mathit{prophs} = (w, v) :: \mathit{prophs}' \twoheadrightarrow \\ \text{proph}\ p\ \gamma\ (\mathit{past} +\!\!+ [(w, v)])\ \mathit{prophs}' \twoheadrightarrow \\ \Phi\ w\end{array}\right\}\end{array}}{\text{WP}\ \texttt{Resolve}\ e\ p\ v\ \ \{\,\Phi\,\}}$$

Prophecy variables with memory

$$\textsc{ProphecyLbGet}\quad \frac{\text{proph } p\ \gamma\ \mathit{past}\ \mathit{prophs}}{\text{proph-lb } \gamma\ \mathit{prophs}}$$

$$\textsc{ProphecyValid}\quad \frac{\text{proph } p\ \gamma\ \mathit{past}\ \mathit{prophs}_1 \qquad \text{proph-lb } \gamma\ \mathit{prophs}_2}{\exists\, \mathit{past}_1, \mathit{past}_2.\ \bigwedge \begin{bmatrix} \mathit{past} = \mathit{past}_1 +\!\!+\ \mathit{past}_2 \\ \mathit{past}_2 +\!\!+\ \mathit{prophs}_1 = \mathit{prophs}_2 \end{bmatrix}}$$

Conclusion

- Coq mechanization is available on `github` :
  `https://github.com/clef-men/caml5`

- Simplified Chase-Lev deque (one infinite array) ✓
  Real-life Chase-Lev deque (multiple circular arrays) 

- Proof looks more complex than the sketch. In particular, transitions between logical states are not really formalized.

- We plan to verify more primitives (Domainslib, Taskflow) based on Chase-Lev deque. This is thanks to modularity of Iris specifications.

Thank you for your attention!

Implementation — chaselev_make

```
let chaselev_make _ =
  let t = AllocN 4 () in
  t.front <- 0 ;
  t.back <- 0 ;
  t.data <- inf_array_make () ;
  t.prophecy <- NewProph ;
  t
```

Implementation — `chaselev_push`

```
let chaselev_push t v =
  let back = !t.back in
  inf_array_set !t.data back v ;
  t.back <- back + 1
```

Implementation — chaselev_steal

```
let rec chaselev_steal t =
  let id = NewId in
  let front = !t.front in
  let back = !t.back in
  if front < back then (
    if Snd (
      Resolve (
        CmpXchg t.front front (front + 1)
       ) !t.prophecy (front, id)
    ) then (
      SOME (inf_array_get !t.data front)
    ) else (
      chaselev_steal t
    )
  ) else (
    NONE
  )
```

Implementation — chaselev_pop

```
let chaselev_pop t =
  let id = NewId in
  let back = !t.back - 1 in
  t.back <- back ;
  let front = !t.front in
  if back < front then (
    t.back <- front
  ) else (
    if front < back then (
      SOME (inf_array_get !t.data back)
    ) else (
      if Snd (
        Resolve (
          CmpXchg t.front front (front + 1)
        ) !t.prophecy (front, id)
      ) then (
        t.back <- front + 1 ;
        SOME (inf_array_get !t.data back)
      ) else (
        t.back <- front + 1 ;
        NONE
      )
```

Infinite array

$$\frac{\{\,\text{True}\,\}}{\texttt{inf\_array\_make}\ v} \over \{\,\lambda\ arr.\,\exists\,\gamma.\,\text{inf-array-inv}\ arr\ \gamma\ \iota * \text{inf-array-model}\ arr\ \gamma\ (\lambda\_.\,v)\,\}$$

$$\begin{array}{c} \{\,\text{inf-array-inv}\ arr\ \gamma\ \iota * 0 \leqslant i\,\} \\ \hline \langle\,\forall\, vs.\,\text{inf-array-model}\ arr\ \gamma\ vs\,\rangle \\ \hline \texttt{inf\_array\_get}\ arr\ i,\ \uparrow\iota \\ \hline \langle\,\exists.\,\text{inf-array-model}\ arr\ \gamma\ vs\,\rangle \\ \hline \{\,vs\ i.\,\text{True}\,\} \end{array}$$

$$\begin{array}{c} \{\,\text{inf-array-inv}\ arr\ \gamma\ \iota * 0 \leqslant i\,\} \\ \hline \langle\,\forall\, vs.\,\text{inf-array-model}\ arr\ \gamma\ vs\,\rangle \\ \hline \texttt{inf\_array\_set}\ arr\ i\ v,\ \uparrow\iota \\ \hline \langle\,\exists.\,\text{inf-array-model}\ arr\ \gamma\ vs[i \mapsto v]\,\rangle \\ \hline \{\,\texttt{()}.\,\text{True}\,\} \end{array}$$

Invariant

$$\text{chaselev-inv } t \ \iota \triangleq$$

$$\exists\, \ell, \gamma, \mathit{data}, p.\ \ast \left[\begin{array}{l} t = \ell \ast \text{meta } \ell \ \gamma \\ \ell.\text{data} \mapsto_{\square} \mathit{data} \ast \ell.\text{prophecy} \mapsto_{\square} p \\ \text{inf-array-inv } \mathit{arr} \ \gamma.\text{data } \iota.\text{data} \\ \boxed{\text{chaselev-inv-inner } \ell \ \gamma \ \iota.\text{inv } \mathit{data} \ p}^{\iota} \end{array}\right.$$

Invariant

$$
\begin{array}{l}
\text{chaselev-inv-inner}\ \ell\ \gamma\ \iota\ \mathit{data}\ p \triangleq \\
\quad \exists\, \mathit{front}, \mathit{back}, \mathit{hist}, \mathit{model}, \mathit{priv}, \mathit{past}, \mathit{prophs}. \\
\quad \ast \left[
\begin{array}{l}
\ell.\text{front} \mapsto \mathit{front} \ast \ell.\text{back} \mapsto \mathit{back} \\
\boxed{\bullet\, (\mathit{back}, \mathit{priv})}^{\gamma.\text{ctl}} \\
\boxed{\bullet\, \mathit{front}}^{\gamma.\text{front}} \\
\text{inf-array-model}\ \mathit{data}\ \gamma.\text{data}\ (\mathit{hist} +\!\!+\ \mathit{model})\ \mathit{priv} \\
\boxed{\bullet\, \mathit{model}}^{\gamma.\text{model}} \ast |\mathit{model}| = (\mathit{back} - \mathit{front})_{+} \\
\text{wise-prophet-model}\ p\ \gamma.\text{prophet}\ \mathit{past}\ \mathit{prophs} \\
\forall (\mathit{front}', \_) \in \mathit{past}.\ \mathit{front}' < \mathit{front} \\
\text{chaselev-state}\ \gamma\ \iota\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{model}\ \mathit{prophs}
\end{array}
\right.
\end{array}
$$

State

$$\text{chaselev-state}\ \gamma\ \iota\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{model}\ \mathit{prophs} \triangleq$$

$$\bigvee \left[\begin{array}{l} \text{chaselev-state}_1\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist} \\ \text{chaselev-state}_2\ \gamma\ \iota\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{model}\ \mathit{prophs} \\ \text{chaselev-lock}\ \gamma * \bigvee \left[\begin{array}{l} \text{chaselev-state}_3\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{prophs} \\ \text{chaselev-state}_4\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist} \end{array}\right. \end{array}\right.$$

State 1 (empty)

$$\text{chaselev-state}_1\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist} \triangleq$$

$$\ast \left[\begin{array}{l} \mathit{front} = \mathit{back} \\ \boxed{\bullet\, \mathit{hist}}^{\gamma.\text{hist}} \ast |\mathit{hist}| = \mathit{front} \\ \boxed{\bullet\, -.\ \circ\, -}^{\gamma.\text{winner}} \end{array}\right.$$

State 2 (non-empty)

$$\text{chaselev-state}_2\ \gamma\ \iota\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{model}\ \mathit{prophs} \triangleq$$

$$\ast \left[\begin{array}{l}
\mathit{front} < \mathit{back} \\
\boxed{\bullet\,(\mathit{hist} + [\mathit{model}[0]])}^{\gamma.\text{hist}} \ast |\mathit{hist}| = \mathit{front} \\
\quad \textbf{match}\ \text{filter}\ (\lambda(\mathit{front}', _).\, \mathit{front}' = \mathit{front})\ \mathit{prophs}\ \textbf{with} \\
\quad |\ [] \Rightarrow \boxed{\bullet - . \circ -}^{\gamma.\text{winner}} \\
\quad |\ (_, \mathit{id}) :: _ \Rightarrow \\
\qquad \bigvee \left[\begin{array}{l}
\boxed{\bullet - . \circ -}^{\gamma.\text{winner}} \\
\text{identifier}\ \mathit{id} \ast \exists\, \Phi.\, \boxed{\bullet\,(\mathit{front}, \Phi)}^{\gamma.\text{winner}} \ast \text{chaselev-au}\ \gamma\ \iota\ \Phi
\end{array}\right.
\end{array}\right.$$

State 3 (emptyish)

$$\text{chaselev-state}_3\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{prophs} \triangleq$$

$$\ast \left[\begin{array}{l} \mathit{front} = \mathit{back} \\ \boxed{\bullet\, \mathit{hist}}^{\gamma.\text{hist}} \ast |\mathit{hist}| = \mathit{front} + 1 \\ \\ \quad \textbf{match}\ \text{filter}\ (\lambda(\mathit{front}', \_).\ \mathit{front}' = \mathit{front})\ \mathit{prophs}\ \textbf{with} \\ \quad \mid [] \Rightarrow \boxed{\circ\, (\mathit{front}, -)}^{\gamma.\text{winner}} \\ \quad \mid \_ \Rightarrow \exists\, \Phi.\ \boxed{\bullet\, (\mathit{front}, \Phi)}^{\gamma.\text{winner}} \ast \Phi\ (\texttt{SOME}\ \mathit{hist}[\mathit{front}]) \end{array} \right.$$

State 4 (super empty)

$$\text{chaselev-state}_4\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist} \triangleq$$

$$\ast \left[\begin{array}{l} \mathit{front} = \mathit{back} + 1 \\ \boxed{\bullet\, \mathit{hist}}^{\gamma.\text{hist}} \ast |\mathit{hist}| = \mathit{front} \\ \boxed{\bullet\, -.\ \circ -}^{\gamma.\text{winner}} \end{array}\right.$$